

Cartier duality for (φ, \hat{G}) -modules

Yoshiyasu Ozeki*

Abstract

In this paper, we prove the Cartier duality for (φ, \hat{G}) -modules which are defined by Tong Liu to classify semistable Galois representations.

1 Introduction

Let k be a perfect field of characteristic $p \geq 2$, $W(k)$ its ring of Witt vectors, $K_0 := W(k)[1/p]$, K a finite totally ramified extension of K_0 , \bar{K} a fixed algebraic closure of K and $G := \text{Gal}(\bar{K}/K)$. Let $r \geq 0$ be an integer. In this paper, we give the Cartier duality for (φ, \hat{G}) -modules. A (φ, \hat{G}) -module is a Kisin module equipped with certain Galois action, which was introduced by Liu in [Li3] to classify lattices in semistable representations of G whose Hodge-Tate weights are in $[0, r]$.

Breuil defined strongly divisible lattices and conjectured that, if $r < p - 1$, the category of strongly divisible lattices of weight r is equivalent to the category $\text{Rep}_{\mathbb{Z}_p}^r(G)$ of G -stable \mathbb{Z}_p -lattices in semistable p -adic representations of G whose Hodge-Tate weights are in $[0, r]$. This conjecture was proved by Liu in [Li2] if $p \geq 3$. To remove the condition $r < p - 1$, Liu defined a free (φ, \hat{G}) -module of height r in [Li3] and proved that there exists a functor \hat{T} from the category of free (φ, \hat{G}) -modules of height r into $\text{Rep}_{\mathbb{Z}_p}^r(G)$ which induces an equivalence of those categories for any $p \geq 2$ and $r \geq 0$. The notion of torsion (φ, \hat{G}) -modules of height r is given in [CL] and there exists a functor \hat{T} from the category of torsion (φ, \hat{G}) -modules of weight r into torsion \mathbb{Z}_p -representations of G as in the case of free (φ, \hat{G}) -modules.

The Cartier duality for Breuil modules and Kisin modules have been studied by Caruso [Ca1] and Liu [Li1], respectively. Let $\hat{\mathfrak{M}}$ be a torsion (resp. free) (φ, \hat{G}) -module of height r . We define the dual $\hat{\mathfrak{M}}^\vee$ of $\hat{\mathfrak{M}}$ in Section 3, which depends on r . Our main result is

Theorem 1.1. *Let $p \geq 2$ be a prime number and $r \geq 0$ an integer. Let $\hat{\mathfrak{M}}$ be a torsion (resp. free) (φ, \hat{G}) -module of height r and $\hat{\mathfrak{M}}^\vee$ its dual.*

- (1) *The dual $\hat{\mathfrak{M}}^\vee$ is a torsion (resp. free) (φ, \hat{G}) -module of height r .*
- (2) *The assignment $\hat{\mathfrak{M}} \mapsto \hat{\mathfrak{M}}^\vee$ is an anti-equivalence on the category of torsion (resp. free) (φ, \hat{G}) -modules and a natural map $\hat{\mathfrak{M}} \rightarrow (\hat{\mathfrak{M}}^\vee)^\vee$ is an isomorphism.*
- (3) *$\hat{T}(\hat{\mathfrak{M}}^\vee) \simeq \hat{T}^\vee(\hat{\mathfrak{M}})(r)$ as \mathbb{Z}_p -representations of G .*

Here $\hat{T}^\vee(\hat{\mathfrak{M}})$ is the dual representation of $\hat{T}(\hat{\mathfrak{M}})$ and the symbol “ (r) ” in the assertion (3) is for the r -th Tate twist.

Now let $p \geq 3$ and $r < p - 1$. Thanks to Liu’s theory of [Li2], we can construct a natural functor $\mathcal{M}_{\hat{\mathcal{R}}}$ from the category of free (φ, \hat{G}) -modules of height r into the category of strongly divisible lattices of weight r , which gives an equivalence of those categories. By using this functor, we obtain the following comparison result between the duality of (φ, \hat{G}) -modules and that of Breuil modules.

*Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan.
e-mail: y-ozeki@math.kyushu-u.ac.jp

Corollary 1.2. *For a free (φ, \hat{G}) -module $\hat{\mathfrak{M}}$, there exists a canonical isomorphism*

$$\mathcal{M}_{\hat{\mathcal{R}}}(\hat{\mathfrak{M}}^\vee) \simeq \mathcal{M}_{\hat{\mathcal{R}}}(\hat{\mathfrak{M}})^\vee.$$

Here $\mathcal{M}_{\hat{\mathcal{R}}}(\hat{\mathfrak{M}})^\vee$ is the Cartier dual of the strongly divisible lattice $\mathcal{M}_{\hat{\mathcal{R}}}(\hat{\mathfrak{M}})$ (cf. [Ca1], Chapter V).

2 Finite \mathbb{Z}_p -representations

2.1 Kisin modules

Let k be a perfect field of characteristic $p \geq 2$, $W(k)$ its ring of Witt vectors, $K_0 := W(k)[1/p]$, K a finite totally ramified extension of K_0 , \bar{K} a fixed algebraic closure of K and $G := \text{Gal}(\bar{K}/K)$. Throughout this paper, we fix a uniformizer $\pi \in K$ and denote by $E(u)$ its Eisenstein polynomial over K_0 . Let $\mathfrak{S} := W(k)[[u]]$ equipped with a Frobenius endomorphism φ via $u \mapsto u^p$ and the natural Frobenius on $W(k)$. A φ -module (over \mathfrak{S}) is a \mathfrak{S} -module \mathfrak{M} equipped with a φ -semilinear map $\varphi: \mathfrak{M} \rightarrow \mathfrak{M}$. A φ -module is called a *Kisin module*. A morphism between two φ -modules $(\mathfrak{M}_1, \varphi_1)$ and $(\mathfrak{M}_2, \varphi_2)$ is a \mathfrak{S} -linear morphism $\mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ compatible with Frobenii φ_1 and φ_2 . Denote by $\text{Mod}_{/\mathfrak{S}}^r$ the category of φ -modules of finite $E(u)$ -height r (or, of height r) in the sense that \mathfrak{M} is of finite type over \mathfrak{S} and the cokernel of φ^* is killed by $E(u)^r$, where φ^* is the \mathfrak{S} -linearization $1 \otimes \varphi: \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{M}$ of φ . Let $\text{Mod}_{/\mathfrak{S}}^{r, \text{tor}}$ be the full subcategory of $\text{Mod}_{/\mathfrak{S}}^r$ consisting of finite \mathfrak{S} -modules \mathfrak{M} which satisfy the following:

- \mathfrak{M} is killed by some power of p ,
- \mathfrak{M} has a two term resolution by finite free \mathfrak{S} -modules, that is, there exists an exact sequence

$$0 \rightarrow \mathfrak{N}_1 \rightarrow \mathfrak{N}_2 \rightarrow \mathfrak{M} \rightarrow 0$$

of \mathfrak{S} -modules where \mathfrak{N}_1 and \mathfrak{N}_2 are finite free \mathfrak{S} -modules.

Let $\text{Mod}_{/\mathfrak{S}}^{r, \text{fr}}$ be the full subcategory of $\text{Mod}_{/\mathfrak{S}}^r$ consisting of finite free \mathfrak{S} -modules. Let $R := \varprojlim \mathcal{O}_{\bar{K}}/p$ where $\mathcal{O}_{\bar{K}}$ is the integer ring of \bar{K} and the transition maps are given by the p -th power map. By the universal property of the ring of Witt vectors $W(R)$ of R , there exists a unique surjective projection map $\theta: W(R) \rightarrow \hat{\mathcal{O}}_{\bar{K}}$ which lifts the projection $R \rightarrow \mathcal{O}_{\bar{K}}/p$ onto the first factor in the inverse limit, where $\hat{\mathcal{O}}_{\bar{K}}$ is the p -adic completion of $\mathcal{O}_{\bar{K}}$. For any integer $n \geq 0$, let $\pi_n \in \bar{K}$ be a p^n -th root of π such that $\pi_{n+1}^p = \pi_n$ and write $\underline{\pi} = (\pi_n)_{n \geq 0} \in R$. Let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative of $\underline{\pi}$. We embed the $W(k)$ -algebra $W(k)[u]$ into $W(R)$ via the map $u \mapsto [\underline{\pi}]$. This embedding extends to an embedding $\mathfrak{S} \hookrightarrow W(R)$, which is compatible with Frobenius endomorphisms.

Denote by $\mathcal{O}_{\mathcal{E}}$ the p -adic completion of $\mathfrak{S}[1/u]$. Then $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with uniformizer p and residue field $k((u))$. Denote by \mathcal{E} the field of fractions of $\mathcal{O}_{\mathcal{E}}$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to inclusions $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\text{Fr}R)$ and $\mathcal{E} \hookrightarrow W(\text{Fr}R)[1/p]$. Here $\text{Fr}R$ is the field of fractions of R . It is not difficult to see that $\text{Fr}R$ is algebraically closed. We denote by \mathcal{E}^{ur} the maximal unramified field extension of \mathcal{E} in $W(\text{Fr}R)[1/p]$ and \mathcal{O}^{ur} its integer ring. Let $\widehat{\mathcal{E}}^{\text{ur}}$ be the p -adic completion of \mathcal{E}^{ur} and $\widehat{\mathcal{O}}^{\text{ur}}$ its integer ring. The ring $\widehat{\mathcal{E}}^{\text{ur}}$ (resp. $\widehat{\mathcal{O}}^{\text{ur}}$) is equal to the closure of \mathcal{E}^{ur} in $W(\text{Fr}R)[1/p]$ (resp. the closure of \mathcal{O}^{ur} in $W(\text{Fr}R)$). Put $\mathfrak{S}^{\text{ur}} := \widehat{\mathcal{O}}^{\text{ur}} \cap W(R)$. We regard all these rings as subrings of $W(\text{Fr}R)[1/p]$. Put $\mathfrak{S}_{\infty}^{\text{ur}} := \mathfrak{S}^{\text{ur}}[1/p]/\mathfrak{S}^{\text{ur}}$.

Let $K_{\infty} := \cup_{n \geq 0} K(\pi_n)$ and $G_{\infty} := \text{Gal}(\bar{K}/K_{\infty})$. Then G_{∞} acts on \mathfrak{S}^{ur} and \mathcal{E}^{ur} continuously and fixes the subring $\mathfrak{S} \subset W(R)$. We denote by $\text{Rep}_{\mathbb{Z}_p}(G_{\infty})$ the category of continuous \mathbb{Z}_p -linear representations of G_{∞} on finite \mathbb{Z}_p -modules. We denote by $\text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(G_{\infty})$ (resp. $\text{Rep}_{\mathbb{Z}_p}^{\text{fr}}(G_{\infty})$) the full subcategory of $\text{Rep}_{\mathbb{Z}_p}(G_{\infty})$ consisting of \mathbb{Z}_p -modules killed by some power of p (resp. finite free \mathbb{Z}_p -modules).

For any $\mathfrak{M} \in \text{Mod}_{/\mathfrak{S}}^{r, \text{tor}}$, we define a $\mathbb{Z}_p[G_\infty]$ -module via

$$T_{\mathfrak{S}}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}_\infty^{\text{ur}}),$$

where a G_∞ -action on $T_{\mathfrak{S}}(\mathfrak{M})$ is given by $(\sigma.g)(x) := \sigma(g(x))$ for $\sigma \in G_\infty, g \in T_{\mathfrak{S}}(\mathfrak{M}), x \in \mathfrak{M}$. The representation $T_{\mathfrak{S}}(\mathfrak{M})$ is an object of $\text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(G_\infty)$.

Similarly, for any $\mathfrak{M} \in \text{Mod}_{/\mathfrak{S}}^{r, \text{fr}}$, we define a $\mathbb{Z}_p[G_\infty]$ -module via

$$T_{\mathfrak{S}}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}_\infty^{\text{ur}}).$$

The representation $T_{\mathfrak{S}}(\mathfrak{M})$ is an object of $\text{Rep}_{\mathbb{Z}_p}^{\text{fr}}(G_\infty)$ and $\text{rank}_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M}) = \text{rank}_{\mathfrak{S}} \mathfrak{M}$.

Theorem 2.1 ([Ki]). *The functor $T_{\mathfrak{S}}: \text{Mod}_{/\mathfrak{S}}^{r, \text{fr}} \rightarrow \text{Rep}_{\mathbb{Z}_p}^{\text{fr}}(G_\infty)$ is fully faithful.*

Proof. The desired assertion follows from Corollary (2.1.4) and Proposition (2.1.12) of [Ki] and Fontaine's theory (below). \square

2.2 Fontaine's theory

A finite $\mathcal{O}_{\mathcal{E}}$ -module M is called *étale* if M is equipped with a φ -semi-linear map $\varphi_M: M \rightarrow M$ such that φ_M^* is an isomorphism, where φ_M^* is the $\mathcal{O}_{\mathcal{E}}$ -linearization $1 \otimes \varphi_M: \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} M \rightarrow M$ of φ_M . We denote by $\Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}}}$ the category of finite étale $\mathcal{O}_{\mathcal{E}}$ -modules with the obvious morphisms. Note that the extension K_∞/K is a strictly APF extension in the sense of [Wi] and thus G_∞ is naturally isomorphic to the absolute Galois group of $k((u))$ by the theory of norm fields. Combining this fact and Fontaine's theory in [Fo], A 1.2.6, we have that the functor

$$T_*: \Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}}} \rightarrow \text{Rep}_{\mathbb{Z}_p}(G_\infty), \quad M \mapsto (\widehat{\mathcal{O}^{\text{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1}$$

is an equivalence of Abelian categories and there exists the following natural $\widehat{\mathcal{O}^{\text{ur}}}$ -linear isomorphism which is compatible with φ -structures and G_∞ -actions:

$$\widehat{\mathcal{O}^{\text{ur}}} \otimes_{\mathbb{Z}_p} T_*(M) \xrightarrow{\sim} \widehat{\mathcal{O}^{\text{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M. \quad (2.2.1)$$

Furthermore, the functor T_* induces equivalences of categories between the category of finite torsion étale $\mathcal{O}_{\mathcal{E}}$ -modules and $\text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(G_\infty)$ (resp. the category of finite free étale $\mathcal{O}_{\mathcal{E}}$ -modules and $\text{Rep}_{\mathbb{Z}_p}^{\text{fr}}(G_\infty)$).

The contravariant version of the functor T_* is useful for integral theory. For any $M \in \Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}}}$, put

$$T(M) := \text{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(M, \mathcal{E}^{\text{ur}}/\mathcal{O}^{\text{ur}}) \quad \text{if } M \text{ is killed by some power of } p$$

and

$$T(M) := \text{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(M, \widehat{\mathcal{O}^{\text{ur}}}) \quad \text{if } M \text{ is } p \text{ torsion free.}$$

Then we can check that $T(M)$ is the dual representation of $T_*(M)$.

Let $\mathfrak{M} \in \text{Mod}_{/\mathfrak{S}}^{r, \text{tor}}$ (resp. $\mathfrak{M} \in \text{Mod}_{/\mathfrak{S}}^{r, \text{fr}}$). Since $E(u)$ is a unit in $\mathcal{O}_{\mathcal{E}}$, we see that $M := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is a finite torsion étale $\mathcal{O}_{\mathcal{E}}$ -module (resp. a finite free étale $\mathcal{O}_{\mathcal{E}}$ -module). Here a Frobenius structure on M is given by $\varphi_M := \varphi_{\mathcal{O}_{\mathcal{E}}} \otimes \varphi_{\mathfrak{M}}$.

Theorem 2.2 ([Li1], Corollary 2.2.2). *Let $\mathfrak{M} \in \text{Mod}_{/\mathfrak{S}}^{r, \text{tor}}$ or $\mathfrak{M} \in \text{Mod}_{/\mathfrak{S}}^{r, \text{fr}}$. Then the natural map*

$$T_{\mathfrak{S}}(\mathfrak{M}) \rightarrow T(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M})$$

is an isomorphism as \mathbb{Z}_p -representations of G_∞ .

2.3 Liu's theory

Let S be the p -adic completion of $W(k)[u, \frac{E(u)^i}{i!}]_{i \geq 0}$ and endow S with the following structures:

- a continuous φ -semilinear Frobenius $\varphi: S \rightarrow S$ defined by $\varphi(u) := u^p$.
- a continuous linear derivation $N: S \rightarrow S$ defined by $N(u) := -u$.
- a decreasing filtration $(\text{Fil}^i S)_{i \geq 0}$ in S . Here $\text{Fil}^i S$ is the p -adic closure of the ideal generated by the divided powers $\gamma_j(E(u)) = \frac{E(u)^j}{j!}$ for all $j \geq i$.

Put $S_{K_0} := S[1/p] = K_0 \otimes_{W(k)} S$. The inclusion $W(k)[u] \hookrightarrow W(R)$ via the map $u \mapsto [\underline{x}]$ induces inclusions $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\text{cris}}$ and $S_{K_0} \hookrightarrow B_{\text{cris}}^+$. We regard all these rings as subrings in B_{cris}^+ .

Fix a choice of primitive p^i -root of unity ζ_{p^i} for $i \geq 0$ such that $\zeta_{p^{i+1}}^p = \zeta_{p^i}$. Put $\underline{\varepsilon} := (\zeta_{p^i})_{i \geq 0} \in R^\times$ and $t := \log([\underline{\varepsilon}]) \in A_{\text{cris}}$. Denote by $\nu: W(R) \rightarrow W(\bar{k})$ a unique lift of the projection $R \rightarrow \bar{k}$. Since $\nu(\text{Ker}(\theta))$ is contained in the set $pW(\bar{k})$, ν extends to a map $\nu: A_{\text{cris}} \rightarrow W(\bar{k})$ and $\nu: B_{\text{cris}}^+ \rightarrow W(\bar{k})[1/p]$. For any subring $A \subset B_{\text{cris}}^+$, we put $I_+ A := \text{Ker}(\nu \text{ on } B_{\text{cris}}^+) \cap A$. For any integer $n \geq 0$, let $t^{\{n\}} := t^{r(n)} \gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p})$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power.

We define a subring \mathcal{R}_{K_0} of B_{cris}^+ as below:

$$\mathcal{R}_{K_0} := \left\{ \sum_{i=0}^{\infty} f_i t^{\{i\}} \mid f_i \in S_{K_0} \text{ and } f_i \rightarrow 0 \text{ as } i \rightarrow \infty \right\}.$$

Put $\hat{\mathcal{R}} := \mathcal{R}_{K_0} \cap W(R)$ and $I_+ := I_+ \hat{\mathcal{R}}$.

For any field F over \mathbb{Q}_p , set $F_{p^\infty} := \bigcup_{n \geq 0} F(\zeta_{p^n})$. Recall $K_\infty = \bigcup_{n \geq 0} K(\pi_n)$ and note that $K_{\infty, p^\infty} = \bigcup_{n \geq 0} K(\pi_n, \zeta_{p^\infty})$ is the Galois closure of K_∞ over K . Put $H_K := \text{Gal}(K_{\infty, p^\infty}/K_\infty)$, $G_{p^\infty} := \text{Gal}(K_{\infty, p^\infty}/K_{p^\infty})$ and $\hat{G} := \text{Gal}(K_{\infty, p^\infty}/K)$.

Proposition 2.3 ([Li3], Lemma 2.2.1). (1) $\hat{\mathcal{R}}$ (resp. \mathcal{R}_{K_0}) is a φ -stable \mathfrak{S} -algebra as a subring in $W(R)$ (resp. B_{cris}^+).

(2) $\hat{\mathcal{R}}$ and I_+ (resp. \mathcal{R}_{K_0} and $I_+ \mathcal{R}_{K_0}$) are G -stable. The G -action on $\hat{\mathcal{R}}$ and I_+ (resp. \mathcal{R}_{K_0} and $I_+ \mathcal{R}_{K_0}$) factors through \hat{G} .

(3) There exist natural isomorphisms $\mathcal{R}_{K_0}/I_+ \mathcal{R}_{K_0} \simeq K_0$ and $\hat{\mathcal{R}}/I_+ \simeq S/I_+ S \simeq \mathfrak{S}/I_+ \mathfrak{S} \simeq W(k)$.

For any Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ of height r , we put $\hat{\mathfrak{M}} := \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ and equip $\hat{\mathfrak{M}}$ with a Frobenius $\varphi_{\hat{\mathfrak{M}}}$ by $\varphi_{\hat{\mathfrak{M}}} := \phi_{\hat{\mathcal{R}}} \otimes \varphi_{\mathfrak{M}}$. It is known that a natural map

$$\mathfrak{M} \rightarrow \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \hat{\mathfrak{M}}$$

is an injection ([CL], Section 3.1). By this injection, we regard \mathfrak{M} as a $\varphi(\mathfrak{S})$ -stable submodule in $\hat{\mathfrak{M}}$.

Definition 2.4. A (φ, \hat{G}) -module (of height r) is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where

- (1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module (of height r),
- (2) \hat{G} is an $\hat{\mathcal{R}}$ -semi-linear \hat{G} -action on $\hat{\mathfrak{M}} := \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$,
- (3) the \hat{G} -action commutes with $\varphi_{\hat{\mathfrak{M}}}$,
- (4) $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$,
- (5) \hat{G} acts on the $W(k)$ -module $\hat{\mathfrak{M}}/I_+ \hat{\mathfrak{M}}$ trivially.

If \mathfrak{M} is a torsion (resp. free) Kisin module of (height r), we call $\hat{\mathfrak{M}}$ a *torsion* (resp. *free*) (φ, \hat{G}) -module (of height r). If $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ is a (φ, \hat{G}) -module, we often abuse of notations by denoting $\hat{\mathfrak{M}}$ the underlying module $\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$.

A morphism $f: (\mathfrak{M}, \varphi, \hat{G}) \rightarrow (\mathfrak{M}', \varphi', \hat{G})$ between two (φ, \hat{G}) -modules is a morphism $f: (\mathfrak{M}, \varphi) \rightarrow (\mathfrak{M}', \varphi')$ of Kisin-modules such that $\hat{\mathcal{R}} \otimes f: \hat{\mathfrak{M}} \rightarrow \hat{\mathfrak{M}}'$ is a \hat{G} -equivalent. We denote by $\text{Mod}_{\mathfrak{S}}^{r, \hat{G}}$ (resp. $\text{Mod}_{\mathfrak{S}}^{r, \hat{G}, \text{tor}}$, resp. $\text{Mod}_{\mathfrak{S}}^{r, \hat{G}, \text{fr}}$) the category of (φ, \hat{G}) -modules (resp. torsion (φ, \hat{G}) -modules, resp. free (φ, \hat{G}) -modules). We regard $\hat{\mathfrak{M}}$ as a G -module via the projection $G \rightarrow \hat{G}$.

For a (φ, \hat{G}) -module $\hat{\mathfrak{M}}$, we define a $\mathbb{Z}_p[G]$ -module as below:

$$\hat{T}(\hat{\mathfrak{M}}) := \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(R)_{\infty}) \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p$$

and

$$\hat{T}(\hat{\mathfrak{M}}) := \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(R)) \quad \text{if } \mathfrak{M} \text{ is free.}$$

Here, $W(R)_{\infty} := W(R)[1/p]/W(R)$ and G acts on $\hat{T}(\hat{\mathfrak{M}})$ by $(\sigma.f)(x) := \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G$, $f \in \hat{T}(\hat{\mathfrak{M}})$, $x \in \hat{\mathfrak{M}}$.

Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be a (φ, \hat{G}) -module. There exists a natural map

$$\theta: T_{\mathfrak{S}}(\mathfrak{M}) \rightarrow \hat{T}(\hat{\mathfrak{M}})$$

defined by

$$\theta(f)(a \otimes m) := a\varphi(f(m)) \quad \text{for } f \in T_{\mathfrak{S}}(\mathfrak{M}), a \in \hat{\mathcal{R}}, m \in \mathfrak{M},$$

which is a G_{∞} -equivalent.

Theorem 2.5 ([Li3], [CL]). *Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be a (φ, \hat{G}) -module.*

- (1) *The map $\theta: T_{\mathfrak{S}}(\mathfrak{M}) \rightarrow \hat{T}(\hat{\mathfrak{M}})$ is an isomorphism of $\mathbb{Z}_p[G_{\infty}]$ -modules.*
- (2) *The functor \hat{T} induces an anti-equivalence between the category $\text{Mod}_{\mathfrak{S}}^{r, \hat{G}, \text{fr}}$ of free (φ, \hat{G}) -modules of height r and the category $\text{Rep}_{\mathbb{Z}_p}^r(G)$ of G -stable \mathbb{Z}_p -lattices in semi-stable p -adic representations of G with Hodge-Tate weights in $[0, r]$.*

3 Cartier duality

Throughout this section, for any \mathbb{Z}_p -module M and integer $n \geq 0$, we put $M_n := \mathbb{Z}_p/p^n \mathbb{Z}_p \otimes_{\mathbb{Z}_p} M$ and $M_{\infty} := \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} M$.

3.1 Duality on Kisin modules

In this subsection, we recall Liu's results on duality theorems for Kisin modules ([Li1], Section 3). We start with the following example:

Example 3.1. Let $\mathfrak{S}^{\vee} := \mathfrak{S} \cdot \mathfrak{f}^r$ be the rank-1 free \mathfrak{S} -module with $\varphi(\mathfrak{f}^r) := c_0^{-r} E(u)^r \cdot \mathfrak{f}^r$ where pc_0 is the constant coefficient of $E(u)$. We denote by φ^{\vee} this Frobenius. Then $(\mathfrak{S}^{\vee}, \varphi^{\vee})$ is a free Kisin module of height r and there exists an isomorphism $T_{\mathfrak{S}}(\mathfrak{S}^{\vee}) \simeq \mathbb{Z}_p(r)$ as $\mathbb{Z}_p[G_{\infty}]$ -modules (see [Li1], Example 2.3.5). Put $\mathfrak{S}_{\infty}^{\vee} := \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\vee} = \mathfrak{S}_{\infty} \cdot \mathfrak{f}^r$ (resp. $\mathfrak{S}_n^{\vee} := \mathbb{Z}_p/p^n \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\vee} = \mathfrak{S}_n \cdot \mathfrak{f}^r$ for any integer $n \geq 0$). The Frobenius φ on \mathfrak{S}^{\vee} induces Frobenii φ^{\vee} on $\mathfrak{S}_{\infty}^{\vee}$ and \mathfrak{S}_n^{\vee} .

Put $\mathcal{E}^{\vee} := \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{S}^{\vee} = \mathcal{E} \cdot \mathfrak{f}^r$ and equip \mathcal{E}^{\vee} with a Frobenius φ^{\vee} arising from that of \mathcal{E} and \mathfrak{S}^{\vee} . Similarly, we put $\mathcal{O}_{\mathcal{E}}^{\vee} = \mathcal{O}_{\mathcal{E}} \cdot \mathfrak{f}^r$, $(\mathcal{E}/\mathcal{O}_{\mathcal{E}})^{\vee} = \mathcal{E}/\mathcal{O}_{\mathcal{E}} \cdot \mathfrak{f}^r$, $\mathcal{O}_{\mathcal{E}, n}^{\vee} = \mathcal{O}_{\mathcal{E}, n} \cdot \mathfrak{f}^r$ and equip them with Frobenii φ^{\vee} which arise from that of \mathcal{E}^{\vee} . We define $\mathcal{O}^{\text{ur}, \vee}$ and $\mathcal{O}_n^{\text{ur}, \vee}$, and Frobenii φ^{\vee} on them by the analogous way.

Let \mathfrak{M} be a Kisin module of height r and denote by $M := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ the corresponding étale φ -module. Put

$$\mathfrak{M}^{\vee} := \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_{\infty}), \quad M^{\vee} := \text{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(M, \mathcal{E}/\mathcal{O}_{\mathcal{E}}) \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p$$

and

$$\mathfrak{M}^{\vee} := \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}), \quad M^{\vee} := \text{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(M, \mathcal{O}_{\mathcal{E}}) \quad \text{if } \mathfrak{M} \text{ is free.}$$

We then have natural pairings

$$\langle \cdot, \cdot \rangle: \mathfrak{M} \times \mathfrak{M}^\vee \rightarrow \mathfrak{S}_\infty^\vee, \quad \langle \cdot, \cdot \rangle: M \times M^\vee \rightarrow (\mathcal{E}/\mathcal{O}_\mathcal{E})^\vee \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p$$

and

$$\langle \cdot, \cdot \rangle: \mathfrak{M} \times \mathfrak{M}^\vee \rightarrow \mathfrak{S}^\vee, \quad \langle \cdot, \cdot \rangle: M \times M^\vee \rightarrow \mathcal{O}_\mathcal{E}^\vee \quad \text{if } \mathfrak{M} \text{ is free.}$$

The Frobenius $\varphi_{\mathfrak{M}}^\vee$ on \mathfrak{M}^\vee (resp. φ_M^\vee on M^\vee) is defined to be

$$\langle \varphi_{\mathfrak{M}}(x), \varphi_{\mathfrak{M}}^\vee(y) \rangle = \varphi^\vee(\langle x, y \rangle) \quad \text{for } x \in \mathfrak{M}, y \in \mathfrak{M}^\vee.$$

$$\text{(resp. } \langle \varphi_M(x), \varphi_M^\vee(y) \rangle = \varphi^\vee(\langle x, y \rangle) \text{ for } x \in M, y \in M^\vee.)$$

Theorem 3.2 ([Li1]). *Let \mathfrak{M} be a torsion (resp. free) Kisin module of height r , $M := \mathcal{O}_\mathcal{E} \otimes_\mathfrak{S} \mathfrak{M}$ the corresponding étale φ -module and $\langle \cdot, \cdot \rangle$ the paring as above.*

(1) *$(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}}^\vee)$ is a torsion (resp. free) Kisin module of height r . Similarly, M^\vee is a torsion (resp. free) étale φ -module.*

(2) *A natural map $\mathcal{O}_\mathcal{E} \otimes_\mathfrak{S} \mathfrak{M}^\vee \rightarrow M^\vee$ is an isomorphism and $\varphi_M^\vee = \varphi_{\mathcal{O}_\mathcal{E}} \otimes \varphi_{\mathfrak{M}}^\vee$.*

(3) *All pairings $\langle \cdot, \cdot \rangle$ appeared in the above are perfect.*

Remark 3.3. The assertion (2) of the above theorem says that there exists a natural isomorphism $\mathcal{O}_\mathcal{E} \otimes_\mathfrak{S} \mathfrak{M}^\vee \simeq (\mathcal{O}_\mathcal{E} \otimes_\mathfrak{S} \mathfrak{M})^\vee = M^\vee$ which is compatible with φ -structures. In fact, the paring $\langle \cdot, \cdot \rangle$ for M is equal to the pairing which is obtained by tensoring $\mathcal{O}_\mathcal{E}$ to the pairing $\langle \cdot, \cdot \rangle$ for \mathfrak{M} .

3.2 Dual (φ, \hat{G}) -modules

In this subsection, we construct dual (φ, \hat{G}) -modules. Put

$$\hat{\mathcal{R}}^\vee := \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}^\vee = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} (\mathfrak{S} \cdot \mathfrak{f}^r) = \hat{\mathcal{R}} \cdot \mathfrak{f}^r,$$

$$\hat{\mathcal{R}}_n^\vee := \mathbb{Z}_p/p^n \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \hat{\mathcal{R}}^\vee = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}_n^\vee = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} (\mathfrak{S}_n \cdot \mathfrak{f}^r) = \hat{\mathcal{R}}_n \cdot \mathfrak{f}^r \quad \text{for any integer } n \geq 0$$

and

$$\hat{\mathcal{R}}_\infty^\vee := \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \hat{\mathcal{R}}^\vee = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}_\infty^\vee = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} (\mathfrak{S}_\infty \cdot \mathfrak{f}^r) = \hat{\mathcal{R}}_\infty \cdot \mathfrak{f}^r,$$

and we equip them with natural Frobenii arising from those of $\hat{\mathcal{R}}$ and \mathfrak{S}^\vee . By Theorem 2.5, we can define a unique \hat{G} -action on $\hat{\mathcal{R}}^\vee$ such that $\hat{\mathcal{R}}^\vee$ has a structure as a (φ, \hat{G}) -module of height r and there exists an isomorphism

$$\hat{T}(\hat{\mathcal{R}}^\vee) \simeq \mathbb{Z}_p(r) \tag{3.2.1}$$

as $\mathbb{Z}_p[G]$ -modules. This \hat{G} -action on $\hat{\mathcal{R}}^\vee$ induces \hat{G} -actions on $\hat{\mathcal{R}}_n^\vee$ and $\hat{\mathcal{R}}_\infty^\vee$. Then it is not difficult to see that $\hat{\mathcal{R}}_n^\vee$ has a structure as a torsion (φ, \hat{G}) -module of height r and there exists an isomorphism

$$\hat{T}(\hat{\mathcal{R}}_n^\vee) \simeq \mathbb{Z}_p/p^n \mathbb{Z}_p(r) \tag{3.2.2}$$

as $\mathbb{Z}_p[G]$ -modules. We may say that $\hat{\mathcal{R}}^\vee$ (resp. $\hat{\mathcal{R}}_n^\vee$) is a dual (φ, \hat{G}) -module of $\hat{\mathcal{R}}$ (resp. $\hat{\mathcal{R}}_n$) since (3.2.1) and (3.2.2) hold.

Remark 3.4. If $K_{p^\infty} \cap K_\infty = K$ (which is automatically hold in the case $p > 2$), then \hat{G} -actions on $\hat{\mathcal{R}}^\vee, \hat{\mathcal{R}}_n^\vee$ and $\hat{\mathcal{R}}_\infty^\vee$ can be written explicitly as follows (see Example 3.2.3 of [Li3]): If $K_{p^\infty} \cap K_\infty = K$, we have $\hat{G} = G_{p^\infty} \rtimes H_K$ (see Lemma 5.1.2 in [Li2]). Fixing a topological generator $\tau \in G_{p^\infty}$, we define \hat{G} -actions on the above three modules by the relation $\tau(\mathfrak{f}^r) := \hat{c}^r \cdot \mathfrak{f}^r$. Here $\hat{c} := \frac{c}{\tau(c)}$, $c := \prod_{n=0}^\infty \varphi^n(\frac{\varphi(c_0^{-1}E(u))}{p})$. Example 3.2.3 of [Li3] says that $c \in A_{\text{cris}}^\times$ and $\hat{c} \in \hat{\mathcal{R}}^\times$. It follows from straightforward calculations that $\hat{\mathcal{R}}^\vee$ and $\hat{\mathcal{R}}_n^\vee$ are (φ, \hat{G}) -modules of height r .

Lemma 3.5. *Let A be a \mathfrak{S} -algebra with characteristic coprime to p . Let \mathfrak{M} be a torsion (resp. free) Kisin module. Then there exist natural isomorphisms:*

$$A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee \xrightarrow{\cong} \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, A_\infty) \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p,$$

$$A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee \xrightarrow{\cong} \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, A) \quad \text{if } \mathfrak{M} \text{ is free.}$$

Proof. If \mathfrak{M} is free, the statement is clear. If $p\mathfrak{M} = 0$, then we may regard \mathfrak{M} as a finite free $\mathfrak{S}/p\mathfrak{S}$ -module and thus the statement is clear. Suppose that \mathfrak{M} is a (general) torsion Kisin module of height r . By Proposition 2.3.2 of [Li1], there exists an extension of φ -modules

$$0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_n = \mathfrak{M}$$

such that, for all $1 \leq i \leq n$, $\mathfrak{M}_i/\mathfrak{M}_{i-1} \in \text{Mod}_{/\mathfrak{S}}^{r, \text{tor}}$ and $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ is a finite free $\mathfrak{S}/p\mathfrak{S} = k[[u]]$ -module. Furthermore, we have $\mathfrak{M}_i \in \text{Mod}_{/\mathfrak{S}}^{r, \text{tor}}$ by Lemma 2.3.1 in [Li1]. We show that the natural map

$$A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i^\vee \longrightarrow \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i, A_\infty), \quad a \otimes f \mapsto (a \otimes x \mapsto af(x))$$

where $a \in A, f \in \mathfrak{M}_i^\vee$ and $x \in \mathfrak{M}_i$, is an isomorphism by induction for i . For $i = 0$, it is obvious. Suppose that the above map is an isomorphism for $i - 1$. We have an exact sequence of \mathfrak{S} -modules

$$0 \rightarrow \mathfrak{M}_{i-1} \rightarrow \mathfrak{M}_i \rightarrow \mathfrak{M}_i/\mathfrak{M}_{i-1} \rightarrow 0. \quad (3.2.3)$$

By Corollary 3.1.5 of [Li1], we know that the sequence

$$0 \rightarrow (\mathfrak{M}_i/\mathfrak{M}_{i-1})^\vee \rightarrow \mathfrak{M}_i^\vee \rightarrow \mathfrak{M}_{i-1}^\vee \rightarrow 0.$$

is also an exact sequence of \mathfrak{S} -modules. Therefore, we have the following exact sequence of A -modules:

$$A \otimes_{\varphi, \mathfrak{S}} (\mathfrak{M}_i/\mathfrak{M}_{i-1})^\vee \rightarrow A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i^\vee \rightarrow A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_{i-1}^\vee \rightarrow 0. \quad (3.2.4)$$

On the other hand, the exact sequence (3.2.3) induces an exact sequence of A -modules

$$0 \rightarrow \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i/\mathfrak{M}_{i-1}, A_\infty) \rightarrow \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i, A_\infty) \rightarrow \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_{i-1}, A_\infty). \quad (3.2.5)$$

Combining sequences (3.2.4) and (3.2.5), we obtain the following commutative diagram of A -modules:

$$\begin{array}{ccccccc} A \otimes_{\varphi, \mathfrak{S}} (\mathfrak{M}_i/\mathfrak{M}_{i-1})^\vee & \longrightarrow & A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i^\vee & \longrightarrow & A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_{i-1}^\vee & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i/\mathfrak{M}_{i-1}, A_\infty) & \longrightarrow & \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i, A_\infty) & \longrightarrow & \text{Hom}_A(A \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_{i-1}, A_\infty) & \end{array}$$

where the two rows are exact. Furthermore, first and third columns are isomorphisms by the induction hypothesis. By the snake lemma, we obtain that the second column is an isomorphism, too. \square

Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be a torsion (resp. free) (φ, \hat{G}) -module of height r and $(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}}^\vee)$ the dual Kisin module of $(\mathfrak{M}, \varphi_{\mathfrak{M}})$. By Lemma 3.5, we have isomorphisms

$$\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee \xrightarrow{\cong} \text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \hat{\mathcal{R}}_\infty^\vee) \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p \quad (3.2.6)$$

(resp.

$$\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee \xrightarrow{\cong} \text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \hat{\mathcal{R}}^\vee) \quad \text{if } \mathfrak{M} \text{ is free}). \quad (3.2.7)$$

We define \hat{G} -action on $\text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \hat{\mathcal{R}}_{\infty}^{\vee})$ (resp. $\text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \hat{\mathcal{R}}^{\vee})$) by

$$(\sigma.f)(x) := \sigma(f(\sigma^{-1}(x)))$$

for $\sigma \in \hat{G}$, $x \in \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ and $f \in \text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \hat{\mathcal{R}}_{\infty}^{\vee})$ (resp. $f \in \text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \hat{\mathcal{R}}^{\vee})$) and equip \hat{G} -action on $\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^{\vee}$ via an isomorphism (3.2.6) (resp. (3.2.7)).

Theorem 3.6. *Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be a torsion (resp. free) (φ, \hat{G}) -module of height r and equip \hat{G} -action on $\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^{\vee}$ as the above. Then the triple $\hat{\mathfrak{M}}^{\vee} := (\mathfrak{M}^{\vee}, \varphi_{\mathfrak{M}}^{\vee}, \hat{G})$ is a torsion (resp. free) (φ, \hat{G}) -module of height r .*

Definition 3.7. We call $\hat{\mathfrak{M}}^{\vee}$ as Theorem 3.6 the *dual* (φ, \hat{G}) -module of $\hat{\mathfrak{M}}$.

To prove Theorem 3.6, we need the following easy property for $\hat{\mathcal{R}}_{\infty} = \hat{\mathcal{R}}[1/p]/\hat{\mathcal{R}}$.

Lemma 3.8. (1) *For any integer n , we have*

$$\hat{\mathcal{R}}[1/p] \cap p^n W(\text{Fr}R) = \hat{\mathcal{R}} \cap p^n W(R) = p^n \hat{\mathcal{R}}.$$

(2) *The following properties for an $a \in \hat{\mathcal{R}}[1/p]$ are equivalent:*

- (i) *If $x \in \hat{\mathcal{R}}[1/p]$ satisfies that $ax = 0$ in $\hat{\mathcal{R}}_{\infty}$, then $x = 0$ in $\hat{\mathcal{R}}_{\infty}$.*
- (ii) *$a \notin p\hat{\mathcal{R}}$.*
- (iii) *$a \notin pW(R)$.*
- (iv) *$a \notin pW(\text{Fr}R)$.*

Proof. (1) The result follows from the relations

$$\hat{\mathcal{R}}[1/p] \cap p^n W(\text{Fr}R) = \hat{\mathcal{R}}[1/p] \cap (W(R)[1/p] \cap p^n W(\text{Fr}R)) = \hat{\mathcal{R}}[1/p] \cap p^n W(R)$$

and

$$p^n \hat{\mathcal{R}} \subset \hat{\mathcal{R}}[1/p] \cap p^n W(R) \subset \hat{\mathcal{R}}_{K_0} \cap p^n W(R) = p^n (\hat{\mathcal{R}}_{K_0} \cap W(R)) = p^n \hat{\mathcal{R}}.$$

(2) The equivalence of (ii), (iii) and (iv) follows from the assertion (1). Suppose the condition (iv) holds. Take any $x \in \hat{\mathcal{R}}[1/p]$ such that $ax \in \hat{\mathcal{R}}$. Then we have

$$\frac{1}{a} \hat{\mathcal{R}} \cap \hat{\mathcal{R}}[1/p] \subset \frac{1}{a} W(\text{Fr}R) \cap W(\text{Fr}R)[1/p] \subset W(\text{Fr}R)$$

since $a \notin pW(\text{Fr}R)$. Thus we obtain

$$x \in \frac{1}{a} \hat{\mathcal{R}} \cap \hat{\mathcal{R}}[1/p] = \frac{1}{a} \hat{\mathcal{R}} \cap \hat{\mathcal{R}}[1/p] \cap W(\text{Fr}R) \subset \hat{\mathcal{R}}[1/p] \cap W(\text{Fr}R) = \hat{\mathcal{R}},$$

which implies the assertion (i) (the last equality follows from (1)). Suppose the condition (ii) does not hold, that is, $a \in p\hat{\mathcal{R}}$. Then $\hat{\mathcal{R}}[1/p] \cap \frac{1}{a} \hat{\mathcal{R}} \supset \frac{1}{a} \hat{\mathcal{R}} \supset \frac{1}{p} \hat{\mathcal{R}} \not\subset \hat{\mathcal{R}}$ and this implies that (i) does not hold. \square

Proof of Theorem 3.6. We only prove the case where $\hat{\mathfrak{M}}$ is a torsion (φ, \hat{G}) -module (the free case can be checked by almost all the same method).

We check the properties (1) to (5) of Definition 2.4 for $\hat{\mathfrak{M}}^{\vee}$. It is clear that (1) and (2) hold for $\hat{\mathfrak{M}}^{\vee}$. Take any $f \in \mathfrak{M}^{\vee}$. Regard \mathfrak{M}^{\vee} as a submodule of $\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^{\vee}$. Then, in $\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^{\vee}$, we see that f is equal to the map

$$\hat{f}: \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \hat{\mathcal{R}} \cdot \mathfrak{f}^r$$

given by $a \otimes x \mapsto a\varphi(f(x)) \cdot \mathfrak{f}^r$ for $a \in \hat{\mathcal{R}}$ and $x \in \mathfrak{M}$. Since $\mathfrak{M} \subset (\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^{H_K}$, we have

$$\begin{aligned} (\sigma.\hat{f})(a \otimes x) &= \sigma(\hat{f}(\sigma^{-1}(a \otimes x))) = \sigma(\hat{f}(\sigma^{-1}(a)(1 \otimes x))) = \sigma((\sigma^{-1}(a)\hat{f}(1 \otimes x))) \\ &= a\sigma(\hat{f}(1 \otimes x)) = a\sigma(\varphi(f(x)) \cdot \mathfrak{f}^r) = a\varphi(f(x)) \cdot \mathfrak{f}^r = \hat{f}(a \otimes x). \end{aligned}$$

for any $a \in \widehat{\mathcal{R}}, x \in \mathfrak{M}$ and $\sigma \in H_K$. This implies $\mathfrak{M}^\vee \subset (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee)^{H_K}$ and hence (4) holds for \mathfrak{M}^\vee . Check the property (5), that is, the condition that \hat{G} acts trivially on $\mathfrak{M}/I_+ \mathfrak{M}$. By Lemma 3.5, we know that there exists the following natural isomorphism:

$$\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee / I_+ (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee) \xrightarrow{\simeq} \text{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} / I_+ (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}), \widehat{\mathcal{R}}_\infty^\vee / I_+ \widehat{\mathcal{R}}_\infty^\vee),$$

which is in fact \hat{G} -equivalent by the definition of \hat{G} -action on $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee$. Since \hat{G} acts on $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} / I_+ (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})$ and $\widehat{\mathcal{R}}_\infty^\vee / I_+ \widehat{\mathcal{R}}_\infty^\vee$ trivially, we obtain the desired result.

Finally we prove the property (3) for \mathfrak{M}^\vee . First we note that, if we take any $f \in \mathfrak{M}^\vee = \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_\infty)$ and regard f as a map which has values in \mathfrak{S}_∞^\vee , then we have

$$\varphi^\vee(f) \circ \varphi_{\mathfrak{M}} = \varphi^\vee \circ f : \mathfrak{M} \rightarrow \mathfrak{S}_\infty^\vee. \quad (3.2.8)$$

Recall that there exists a natural isomorphism

$$\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee \simeq \text{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \widehat{\mathcal{R}}_\infty^\vee)$$

by Lemma 3.5. We equip $\text{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \widehat{\mathcal{R}}_\infty^\vee)$ with a φ -structure φ^\vee via this isomorphism. Then it is enough to show that $\sigma \varphi^\vee = \varphi^\vee \sigma$ on $\text{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \widehat{\mathcal{R}}_\infty^\vee)$ for any $\sigma \in \hat{G}$. Take any $\hat{f} \in \text{Hom}(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \widehat{\mathcal{R}}_\infty^\vee)$ and consider the following diagram:

$$\begin{array}{ccc} \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} & \xrightarrow{\varphi_{\mathfrak{M}}} & \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \\ \hat{f} \downarrow & & \downarrow \varphi^\vee(\hat{f}) \\ \widehat{\mathcal{R}}_\infty^\vee & \xrightarrow{\varphi^\vee} & \widehat{\mathcal{R}}_\infty^\vee. \end{array} \quad (3.2.9)$$

By (3.2.8), we obtain that the diagram (3.2.9) is also commutative. To check the relation $\sigma(\varphi^\vee(\hat{f})) = \varphi^\vee(\sigma(\hat{f}))$, it suffices to show that $\sigma(\varphi^\vee(\hat{f}))(\varphi_{\mathfrak{M}}(x)) = \varphi^\vee(\sigma(\hat{f}))(\varphi_{\mathfrak{M}}(x))$ for any $x \in \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ since \mathfrak{M} is of finite $E(u)$ -height and, for any $a \in \widehat{\mathcal{R}}_\infty$, $\varphi(E(u))a = 0$ if and only if $a = 0$ by Lemma 3.8. By (3.2.9), we have

$$\sigma(\varphi^\vee(\hat{f}))(\varphi_{\mathfrak{M}}(x)) = \sigma(\varphi^\vee(\hat{f})(\sigma^{-1}(\varphi_{\mathfrak{M}}(x)))) = \sigma(\varphi^\vee(\hat{f})(\varphi_{\mathfrak{M}}(\sigma^{-1}(x)))) = \sigma(\varphi^\vee(\hat{f}(\sigma^{-1}(x)))).$$

By replacing \hat{f} with $\sigma(\hat{f})$ in the diagram (3.2.9), we have

$$\varphi^\vee(\sigma(\hat{f}))(\varphi_{\mathfrak{M}}(x)) = \varphi^\vee(\sigma(\hat{f}))(x) = \varphi^\vee(\sigma(\hat{f}(\sigma^{-1}(x))) = \sigma(\varphi^\vee(\hat{f}(\sigma^{-1}(x))))$$

and this finishes the proof. \square

Corollary 3.9. *Let $\hat{\mathfrak{M}}$ be a (φ, \hat{G}) -module. Then the natural map of (φ, \hat{G}) -modules*

$$\hat{\mathfrak{M}} \rightarrow (\hat{\mathfrak{M}}^\vee)^\vee, \quad x \in \hat{\mathfrak{M}} \mapsto (f \mapsto f(x)) \in (\hat{\mathfrak{M}}^\vee)^\vee \quad (3.2.10)$$

is an isomorphism of (φ, \hat{G}) -modules.

Proof. It is known that the map (3.2.10) is an isomorphism by Proposition 3.1.7 in [Li1]. Hence it is enough to check that the map (3.2.10) is compatible with Galois action after tensoring $\widehat{\mathcal{R}}$, but it follows from straightforward calculations. \square

Since the assignment $\hat{\mathfrak{M}} \mapsto \hat{\mathfrak{M}}^\vee$ is a functor from the category of torsion (resp. free) (φ, \hat{G}) -modules of height r to itself, we obtain the following.

Corollary 3.10. *The assignment $\hat{\mathfrak{M}} \mapsto \hat{\mathfrak{M}}^\vee$ is an anti-equivalence on the category of torsion (resp. free) (φ, \hat{G}) -modules. A quasi-inverse is given by $\hat{\mathfrak{M}} \mapsto \hat{\mathfrak{M}}^\vee$.*

3.3 Compatibility with Galois actions

The goal of this subsection is to prove the following:

Theorem 3.11. *Let $\hat{\mathfrak{M}}$ be a (φ, \hat{G}) -module. Then we have*

$$\hat{T}(\hat{\mathfrak{M}}^\vee) \simeq \hat{T}^\vee(\hat{\mathfrak{M}})(r) \quad (3.3.1)$$

as $\mathbb{Z}_p[G]$ -modules where $\hat{T}^\vee(\hat{\mathfrak{M}})$ is the dual representation of $\hat{T}(\hat{\mathfrak{M}})$ and the symbol “ (r) ” is for the r -th Tate twist.

First we construct a covariant functor for the category of (φ, \hat{G}) -modules. Recall that, if $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ is a (φ, \hat{G}) -module, we often abuse of notations by denoting $\hat{\mathfrak{M}}$ the underlying module $\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$.

Proposition 3.12. *Let $\hat{\mathfrak{M}}$ be a (φ, \hat{G}) -module. Then the natural $W(\text{Fr}R)$ -linear map*

$$W(\text{Fr}R) \otimes_{\mathbb{Z}_p} (W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1} \rightarrow W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}, \quad a \otimes x \mapsto ax, \quad (3.3.2)$$

for any $a \in W(\text{Fr}R)$ and $x \in (W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1}$, is an isomorphism, which is compatible with φ -structures and G -actions.

Proof. A non-trivial assertion of this proposition is only the bijectivity of the map (3.3.2). First we note the following natural φ -equivariant isomorphisms:

$$\begin{aligned} W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} &\simeq W(\text{Fr}R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \\ &\simeq W(\text{Fr}R) \otimes_{\mathcal{O}_{\mathcal{E}}} (\mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathfrak{S}} M) \\ &\xrightarrow{1 \otimes \varphi_M^*} W(\text{Fr}R) \otimes_{\mathcal{O}_{\mathcal{E}}} M \end{aligned}$$

where $M := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is the étale φ -module corresponding to \mathfrak{M} . Here the bijectivity of $1 \otimes \varphi_M^*$, where φ_M^* is the $\mathcal{O}_{\mathcal{E}}$ -linearization of φ_M , follows from the étaleness of M . Combining the above isomorphisms and the relation (2.2.1), we obtain the following natural φ -equivalent bijective maps

$$W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} \xrightarrow{\simeq} W(\text{Fr}R) \otimes_{\mathcal{O}_{\mathcal{E}}} M \xleftarrow{\simeq} W(\text{Fr}R) \otimes_{\mathbb{Z}_p} (\mathcal{O}^{\text{ur}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1} \quad (3.3.3)$$

and hence we obtain

$$(W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1} \simeq (\mathcal{O}^{\text{ur}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1}. \quad (3.3.4)$$

By (3.3.3) and (3.3.4), we obtain an isomorphism

$$W(\text{Fr}R) \otimes_{\mathbb{Z}_p} (W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1} \xrightarrow{\simeq} W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}$$

and the desired result follows from the fact that this isomorphism coincides with the natural map (3.3.2). \square

For any (φ, \hat{G}) -module $\hat{\mathfrak{M}}$, we set

$$\hat{T}_*(\hat{\mathfrak{M}}) := (W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1}.$$

Since the Frobenius action on $W(\text{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}$ commutes with G -action, we see that G acts on $\hat{T}_*(\hat{\mathfrak{M}})$ stable. We have shown in the proof of Proposition 3.12 (see (3.3.4)) that

$$\hat{T}_*(\hat{\mathfrak{M}}) \simeq T_*(M)$$

as $\mathbb{Z}_p[G_{\infty}]$ -modules for $M := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ (recall that the functor T_* is defined in Section 2.2). In particular, if $\hat{\mathfrak{M}}$ is free and $d := \text{rank}_{\mathfrak{S}}(\mathfrak{M})$, $\hat{T}_*(\hat{\mathfrak{M}})$ is free of rank d as a \mathbb{Z}_p -module. The association $\hat{\mathfrak{M}} \mapsto \hat{T}_*(\hat{\mathfrak{M}})$ is a covariant functor from the category of (φ, \hat{G}) -modules of height r to the category $\text{Rep}_{\mathbb{Z}_p}(G)$ of finite $\mathbb{Z}_p[G]$ -modules. By the exactness of the functor T_* , the functor \hat{T}_* is an exact functor.

Corollary 3.13. *The \mathbb{Z}_p -representation $\hat{T}_*(\hat{\mathfrak{M}})$ of G is the dual of $\hat{T}(\hat{\mathfrak{M}})$, that is,*

$$\hat{T}^\vee(\hat{\mathfrak{M}}) \simeq \hat{T}_*(\hat{\mathfrak{M}})$$

as $\mathbb{Z}_p[G]$ -modules.

Proof. Suppose $\hat{\mathfrak{M}}$ is killed by some power of p . By Proposition 3.12 and the relation $W(\mathrm{Fr}R)_\infty^{\varphi=1} = \mathbb{Q}_p/\mathbb{Z}_p$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}_*(\hat{\mathfrak{M}}), \mathbb{Q}_p/\mathbb{Z}_p) &\simeq \mathrm{Hom}_{W(\mathrm{Fr}R), \varphi}(W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (W(\mathrm{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1}, W(\mathrm{Fr}R)_\infty) \\ &\simeq \mathrm{Hom}_{W(\mathrm{Fr}R), \varphi}(W(\mathrm{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}, W(\mathrm{Fr}R)_\infty) \\ &\simeq \mathrm{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(\mathrm{Fr}R)_\infty) = \hat{T}(\hat{\mathfrak{M}}). \end{aligned}$$

The last equality follows from the proof of Lemma 3.1.1 of [Li3], but we include a proof here for the sake of completeness. Take any $h \in \mathrm{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(\mathrm{Fr}R)_\infty)$. It is enough to prove that h has in fact values in $W(R)_\infty$. Put $g := h|_{\mathfrak{M}}$. Since g is a $\varphi(\mathfrak{S})$ -linear morphism from \mathfrak{M} to $W(R)_\infty = \varphi(W(R)_\infty)$, there exists a \mathfrak{S} -linear morphism $\mathbf{g}: \mathfrak{M} \rightarrow W(\mathrm{Fr}R)_\infty$ such that $\varphi(\mathbf{g}) = g$. Furthermore, we see that \mathbf{g} is φ -equivariant. Note that $\mathbf{g}(\mathfrak{M}) \subset W(\mathrm{Fr}R)_\infty$ is a \mathfrak{S} -finite type φ -stable submodule and of $E(u)$ -height r . By [Fo], Proposition B.1.8.3, we have $\mathbf{g}(\mathfrak{M}) \subset \mathfrak{S}^{\mathrm{ur}}$. Since

$$h(a \otimes x) = a\varphi(\mathbf{g}(x))$$

for any $a \in \hat{\mathcal{R}}$ and $x \in \mathfrak{M}$, we obtain that h has values in $W(R)_\infty$.

The case $\hat{\mathfrak{M}}$ is free, we obtain the desired result by the same proof as above if we replace $W(\mathrm{Fr}R)_\infty$ (resp. $\mathbb{Q}_p/\mathbb{Z}_p$) with $W(\mathrm{Fr}R)$ (resp. \mathbb{Z}_p). \square

In the rest of this subsection, we prove Theorem 3.11. We only prove the case where \mathfrak{M} is killed by p^n for some integer $n \geq 1$ (we can prove the free case by an analogous way and the free case is easier than the torsion case).

First we consider natural pairings

$$\langle \cdot, \cdot \rangle: \mathfrak{M} \times \mathfrak{M}^\vee \rightarrow \mathfrak{S}_n^\vee \quad (3.3.5)$$

and

$$\langle \cdot, \cdot \rangle: M \times M^\vee \rightarrow \mathcal{O}_{\mathcal{E}, n}^\vee \quad (3.3.6)$$

which are perfect and compatible with φ -structures. Here $M := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is the étale φ -module corresponding to \mathfrak{M} . We can extend the pairing (3.3.6) to the φ -equivalent perfect pairing

$$(\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M) \times (\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M^\vee) \rightarrow \mathcal{O}_n^{\mathrm{ur}, \vee}.$$

Since the above pairing is φ -equivariant and $(\mathcal{O}_n^{\mathrm{ur}, \vee})^{\varphi=1} \simeq \mathbb{Z}_p/p^n\mathbb{Z}_p(-r)$, we have a pairing

$$(\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1} \times (\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M^\vee)^{\varphi=1} \rightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p(-r) \quad (3.3.7)$$

compatible with G_∞ -actions. Liu showed in the proof of Lemma 3.1.2 in [Li1] that this pairing is perfect. By similar way, we have the following pairing

$$(W(\mathrm{Fr}R) \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1} \times (W(\mathrm{Fr}R) \otimes_{\mathcal{O}_{\mathcal{E}}} M^\vee)^{\varphi=1} \rightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p(-r). \quad (3.3.8)$$

On the other hand, the pairing (3.3.5) induces a pairing

$$(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) \times (\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee) \rightarrow \hat{\mathcal{R}}_n^\vee. \quad (3.3.9)$$

We can extend the pairing (3.3.9) to the φ -equivalent perfect pairing

$$(W(\mathrm{Fr}R) \otimes_{\hat{\mathcal{R}}} (\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})) \times (W(\mathrm{Fr}R) \otimes_{\hat{\mathcal{R}}} (\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee)) \rightarrow W(\mathrm{Fr}R) \otimes_{\hat{\mathcal{R}}} \hat{\mathcal{R}}_n^\vee.$$

Since the above pairing is φ -equivariant and $(W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathcal{R}}_n^\vee)^{\varphi=1} \simeq \mathbb{Z}_p/p^n \mathbb{Z}_p(-r)$, we have a pairing

$$(W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}))^{\varphi=1} \times (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee))^{\varphi=1} \rightarrow \mathbb{Z}_p/p^n \mathbb{Z}_p(-r) \quad (3.3.10)$$

compatible with G -actions. Since we have the natural isomorphism $\widehat{\mathcal{O}}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} (\widehat{\mathcal{O}}^{\mathrm{ur}} \otimes_{\mathcal{O}_\varepsilon} M)^{\varphi=1} \xrightarrow{\simeq} \widehat{\mathcal{O}}^{\mathrm{ur}} \otimes_{\mathcal{O}_\varepsilon} M$, we obtain the φ -equivariant isomorphisms

$$W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{M}} \xrightarrow{\simeq} W(\mathrm{Fr}R) \otimes_{\mathcal{O}_\varepsilon} M \xleftarrow{\simeq} W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (\widehat{\mathcal{O}}^{\mathrm{ur}} \otimes_{\mathcal{O}_\varepsilon} M)^{\varphi=1}. \quad (3.3.11)$$

Therefore, combining (3.3.7), (3.3.8), (3.3.10) and (3.3.11), we have the following diagram

$$\begin{array}{ccccc} (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{M}})^{\varphi=1} & \times & (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{M}}^\vee)^{\varphi=1} & \longrightarrow & \mathbb{Z}_p/p^n \mathbb{Z}_p(-r) \\ \simeq \downarrow & & \simeq \downarrow & & \parallel \\ (W(\mathrm{Fr}R) \otimes_{\mathcal{O}_\varepsilon} M)^{\varphi=1} & \times & (W(\mathrm{Fr}R) \otimes_{\mathcal{O}_\varepsilon} M^\vee)^{\varphi=1} & \longrightarrow & \mathbb{Z}_p/p^n \mathbb{Z}_p(-r) \\ \simeq \uparrow & & \simeq \uparrow & & \parallel \\ (\widehat{\mathcal{O}}^{\mathrm{ur}} \otimes_{\mathcal{O}_\varepsilon} M)^{\varphi=1} & \times & (\widehat{\mathcal{O}}^{\mathrm{ur}} \otimes_{\mathcal{O}_\varepsilon} M^\vee)^{\varphi=1} & \longrightarrow & \mathbb{Z}_p/p^n \mathbb{Z}_p(-r) \end{array}$$

It is a straightforward calculation to check that the above diagram is commutative. Since the bottom pairing is perfect, we see that the top pairing is also perfect. This implies $\hat{T}_*(\widehat{\mathfrak{M}}^\vee) \simeq \hat{T}_*(\widehat{\mathfrak{M}})(-r)$ and therefore, we have the desired result by Corollary 3.13.

Remark 3.14. A triple $\widehat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ is called a *weak (φ, \hat{G}) -module* if it only satisfies axioms (1), (2), (3) and (4) in Definition 2.4. Weak (φ, \hat{G}) -modules are related with potentially semistable representations. By the same proofs, all results in this section hold if we replace “ (φ, \hat{G}) -modules” with “weak (φ, \hat{G}) -modules”.

3.4 Comparisons with Breuil modules

Throughout this subsection, we suppose $p \geq 3$ and $r < p - 1$. In this situation, Liu showed in [Li2] that there exists a contravariant functor T_{st} from the category $\mathrm{Mod}_{/S}^r$ of strongly divisible lattices of weight r into the category $\mathrm{Rep}_{\mathbb{Z}_p}^r(G)$ which gives an equivalence of those categories.

Hence there exists an equivalence of categories between $\mathrm{Mod}_{/S}^r$ and $\mathrm{Mod}_{/\mathfrak{S}}^{r, \hat{G}, \mathrm{fr}}$ (see also Theorem 2.5 (2)). Liu’s arguments in Section 5 of [Li2] give an explicit correspondence between the objects of those categories as below.

Let $\widehat{\mathfrak{M}}$ be a free (φ, \hat{G}) -module of height r and put $T := \hat{T}(\widehat{\mathfrak{M}})$. Then $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a semistable p -adic representation of G with Hodge-Tate weights in $[0, r]$ and $\mathcal{D} := S_{K_0} \otimes_{\varphi, \mathfrak{S}} \widehat{\mathfrak{M}}$ has a structure as a Breuil module corresponding¹ to V . If we put $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \widehat{\mathfrak{M}} \subset \mathcal{D}$, then \mathcal{M} has a structure as a quasi-strongly divisible lattice in \mathcal{D} . Liu showed in Lemma 3.5.3 of [Li2] that \mathcal{M} is in fact automatically stable under the monodromy operator $N_{\mathcal{D}}$ of the Breuil module \mathcal{D} and thus \mathcal{M} has a structure as a strongly divisible lattice in \mathcal{D} . Moreover, $T_{\mathrm{st}}(\mathcal{M})$ is isomorphic to T . On the other hands, we can realize the monodromy operator $N_{\mathcal{D}}$ via the action of τ on $\widehat{\mathfrak{M}}$. To see this, we define a natural G -action on $B_{\mathrm{cris}}^+ \otimes_{S_{K_0}} \mathcal{D}$ as follows: For any $\sigma \in G$ and $a \otimes x \in B_{\mathrm{cris}}^+ \otimes_{S_{K_0}} \mathcal{D}$, define

$$\sigma(a \otimes x) = \sum_{i=0}^{\infty} \sigma(a) \gamma_i(-\log([\underline{\varepsilon}(\sigma)])) \otimes N_{\mathcal{D}}^i(x),$$

¹ Breuil showed, in Théorème 6.1.1 of [Br], an equivalence of categories between the category of semistable representations of G and the category of Breuil modules.

where $\underline{\varepsilon}(\sigma) := (\sigma(\pi_n)/\pi_n)_{n \geq 0} \in R$. This action is a well-defined B_{cris}^+ -semi-linear G -action on $B_{\text{cris}}^+ \otimes_{S_{K_0}} \mathcal{D}$ ([Li2], Lemma 5.1.1). From now on, we fix $t := -\log([\underline{\varepsilon}(\tau)])$. Then, for any $n \geq 0$ and $x \in \mathcal{D}$, an induction on n shows that

$$(\tau - 1)^n(x) = \sum_{m=n}^{\infty} \left(\sum_{i_1+i_2+\dots+i_n=m, i_j \geq 1} \frac{m!}{i_1! \dots i_n!} \right) \gamma_m(t) \otimes N_{\mathcal{D}}^m(x).$$

Hence if we put $\log(\tau)(x) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau-1)^n}{n}(x)$ for any $x \in \mathcal{D}$, we have

$$\log(\tau)(x) = t \otimes N_{\mathcal{D}}(x).$$

Therefore, we obtain the well-defined functor

$$\mathcal{M}_{\widehat{\mathcal{R}}}: \text{Mod}_{/\mathfrak{S}}^{r, \hat{G}, \text{fr}} \rightarrow \text{Mod}_{/S}^r, \quad \hat{\mathfrak{M}} \rightarrow S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$$

which makes the following diagram commutative:

$$\begin{array}{ccc} \text{Mod}_{/\mathfrak{S}}^{r, \hat{G}, \text{fr}} & \xrightarrow[\hat{T}]{\simeq} & \text{Rep}_{\mathbb{Z}_p}^r(G) \\ \mathcal{M}_{\widehat{\mathcal{R}}} \downarrow & & \parallel \\ \text{Mod}_{/S}^r & \xrightarrow[T_{\text{st}}]{\simeq} & \text{Rep}_{\mathbb{Z}_p}^r(G). \end{array}$$

Here we equip $\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}}) = S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ with the following additional structures; an S -submodule $\text{Fil}^r(\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}}))$ of $\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})$ defined by

$$\text{Fil}^r(\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})) := \{x \in \mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}}) \mid (1 \otimes \varphi)(x) \in \text{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}\},$$

a φ_S -semi-linear endomorphism $\varphi_r: \text{Fil}^r(\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})) \rightarrow \mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})$ defined by the composition

$$\text{Fil}^r(\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})) \xrightarrow{1 \otimes \varphi} \text{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\frac{1}{p^r} \varphi \otimes 1} S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})$$

and a monodromy operator N on $\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})$ given by $N = \frac{1}{t} \log(\tau)$. Since the above diagram is commutative, we see that the functor $\mathcal{M}_{\widehat{\mathcal{R}}}$ gives an equivalence of categories. By the Galois compatibility of a duality (cf. Theorem 3.11), we obtain

Corollary 3.15. *Let $\hat{\mathfrak{M}}$ be a free (φ, \hat{G}) -module. Then there exists an isomorphism*

$$\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}}^\vee) \simeq \mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})^\vee,$$

which is functorial for $\hat{\mathfrak{M}}$. Here $\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})^\vee$ is the Cartier dual of the strongly divisible lattice $\mathcal{M}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}})$ (cf. [Ca1], Chapter V).

References

- [Br] Christophe Breuil, *Représentations p -adiques semi-stables et transversalité de Griffiths*, Math. Ann. **307**, 191–224 (1997).
- [Ca1] Xavier Caruso, *Conjecture de l’inertie modre de Serre*, Thesis, Université de Paris-Sud 11 (2005).
- [Ca2] Xavier Caruso, *Représentations semi-stables de torsion dans le case $er < p - 1$* , J. Reine Angew. Math. **594**, 35–92 (2006).

- [CL] Xavier Caruso and Tong Liu, *Some bounds for ramification of p^n -torsion semistable representations*.
- [Fo] Jean-Marc Fontaine, *Représentations p -adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. **87**, Birkhäuser Boston, Boston, MA 61–88 (2008).
- [Ki] Mark Kisin, *Crystalline representations and F -crystals*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 459–496 (2006).
- [Li1] Tong Liu, *Torsion p -adic Galois representations and a conjecture of Fontaine*, Ann. Sci. École Norm. Sup. (4) **40**, no.4, 633–674 (2007).
- [Li2] Tong Liu, *On lattices in sem-stable representations: A proof of a conjecture of Breuil*, Compositio Math. **144**(1), 61–88 (2008).
- [Li3] Tong Liu, *A note on lattices in semi-stable representations*, Math. Ann. **346** 117–138 (2010).
- [Wi] Jean-Pierre Wintenberger, *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. École Norm. Sup. (4), vol. **16**, 59–89 (1983).